Quantization of the Dirac field

$$S = \int d^4x \; \bar{\psi}(i\not{\partial} - m)\psi = \int d^4x \left[\frac{i}{2}\left(\bar{\psi}\not{\partial}\psi - \partial_\mu \bar{\psi}\gamma^\mu \psi\right) - m\bar{\psi}\psi \right]$$

- $$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)_\alpha} \partial_\nu \psi^\alpha + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \partial_\nu \bar{\psi} - \delta^\mu_\nu \mathcal{L}$$

$$= \frac{i}{2}\left(\bar{\psi}\gamma^\mu \partial_\nu \psi - \partial_\nu \bar{\psi}\gamma^\mu \psi\right) - \delta^\mu_\nu \mathcal{L}$$

$$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

- $$H = \int d^3x \; T^0{}_0 = \int d^3x \; \bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi$$

$$H = P^0 = \int d^3x \; \psi^\dagger h_D \psi$$

$$h_D = \gamma^0\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right) \equiv \text{"Dirac Hamiltonian"}$$

- $$P_i = \int T^0_{\ i} = \frac{i}{2}\int \bar{\psi}\gamma^0\partial_i\psi - \partial_i\bar{\psi}\gamma^0\psi = i\int \psi^\dagger\partial_i\psi$$

$$P^i = -P_i = -i\int \psi^\dagger\partial_i\psi = \int d^3x\, \psi^\dagger\, -i\vec{\nabla}\,\psi$$

$$\partial_\mu$$

$$= \int d^3x\, \frac{-i}{2}\left(\psi^\dagger\vec{\nabla}\psi - (\vec{\nabla}\psi^\dagger)\psi\right) \quad \vec{p} = -i\vec{\nabla}$$

- Notice $\quad H = P^0 = \int d^3x\, \psi^\dagger h_D\psi \overset{\text{on-shell}}{=} \frac{i}{2}\int \psi^\dagger\dot{\psi} - \dot{\psi}^\dagger\psi$

Canonical variables

- tentative $\mathcal{L} = \frac{i}{2}\left(\bar{\psi}\gamma^\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma^\mu \psi\right) - m\bar{\psi}\psi$

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = \frac{i}{2}\left(\bar{\psi}\gamma^0\right)_\alpha = \frac{i}{2}\psi^+_\alpha$$

$$\pi^+_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{\psi}^+_\alpha} = -\frac{i}{2}\psi_\alpha$$

- ψ_α et ψ^+_α are canonically conjugated !

- $\mathcal{L}$ contains $\dot{\psi}^+_\alpha \propto \dot{\pi}_\alpha$

$$\int \dot{\varphi}^2 - (\vec{\nabla}\varphi)^2$$

$$+\frac{\dot{\varphi}^2}{2} - \frac{1}{2}\varphi\ddot{\varphi} - (\vec{\nabla}\varphi)^2$$

$$\left\{ \psi_\alpha(x), \; i \frac{\psi_\beta^\dagger(y)}{2} \right\} = i\, \delta(x-y)\, \delta_{\alpha\beta}$$

$$\left\{ \psi_\beta^\dagger(x), \; -i \frac{\psi_\alpha(y)}{2} \right\} = i\, \delta(x-y)\, \delta_{\alpha\beta}$$

- better work in complex form:

$$\mathcal{L} = i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \implies \boxed{\Pi_\alpha = i\psi^\dagger_\alpha}$$

<u>quantize:</u>

$$[\psi_\alpha(x), \psi^\dagger_\beta(y)] = \delta^3(x-y)\,\delta_{\alpha\beta}$$

A) $$[\psi_\alpha(x), \Pi_\beta(y)] = [\psi_\alpha(x), i\psi^\dagger_\beta(y)] = i\delta^3(x-y)\,\delta_{\alpha\beta}$$

$$\hookrightarrow \quad [\psi_\alpha(x), \psi^\dagger_\beta(y)] = \delta^3(x-y)\,\delta_{\alpha\beta}$$

B) $$[\psi_\alpha(x), \psi_\beta(y)] = [\psi^\dagger_\alpha(x), \psi^\dagger_\beta(y)] = 0$$

- $\dot{\psi} = i[H, \psi] \sim$ Dirac eq.

$$i[H, \psi_\alpha(x)] = i\int d^3y \, [\psi^\dagger(y)\, h_D \psi(y), \psi_\alpha(x)]$$

$$= i\int d^3y \, [\psi^\dagger_\beta(y), \psi_\alpha(x)\} \, (h_D \psi(y))_\beta$$

$$\dot{\psi}_\alpha(x) = -i\,(h_D \psi(x))_\alpha = -i\left[\gamma^0(-i\,\vec{\gamma}\cdot\vec{D} + m)\psi\right]_\alpha$$

$$\Rightarrow \quad \gamma^0 \dot{\psi} = -(\vec{\gamma}\cdot\vec{D} + im)\psi \quad \Rightarrow \quad (i\gamma^\mu \partial_\mu - m)\psi = 0$$

Hilbert Space

$$H = \int \psi^{\dagger} h_D \psi \, d^3x$$

- expand ψ in eigenstates of $h_D \Longrightarrow u_s(\vec{p}),\ v_s(\vec{p})$

$$\left(i\gamma^0 \partial_t + i\vec{\gamma}\cdot\vec{\nabla} - m\right) e^{-i(\omega_p t - \vec{p}\cdot\vec{x})} u_s(\vec{p}) = 0$$

$$\Longrightarrow \quad h_D u_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} = \omega_p u_s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} \xrightarrow{\text{eigenvalue}} \omega_p$$

positive energy solution

$$\left(i\gamma^0 \partial_t + i\vec{\gamma}\cdot\vec{\nabla} - m\right) e^{+i(\omega_p t - \vec{p}\cdot\vec{x})} v_s(\vec{p}) = 0$$

$$\Longrightarrow \quad h_D v_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} = -\omega_p v_s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \xrightarrow{\text{eigenvalue}} -\omega_p$$

negative energy solution

numerical spinors

$$\Longrightarrow \psi(x) = \int d\Omega_p \, e^{i\vec{p}\cdot\vec{x}} \sum_{s=1}^{2} \left[a_s(\vec{p})\, u_s(\vec{p}) + b_s(-\vec{p})\, v_s(-\vec{p}) \right]$$

operator   operator

- conversely

$$\int e^{-i\vec{p}\vec{x}}\, \psi(x)\, d^3x = \frac{1}{2\omega_p} \sum_{s=1}^{2} \left[a_s(\vec{p})\, u_s(\vec{p}) + b_s(-\vec{p})\, v_s(-\vec{p}) \right]$$

$$\int e^{-i\vec{p}\cdot\vec{x}}\, \bar{u}_s(\vec{p})\, \gamma^0 \psi(x) = a_s(\vec{p})$$

$$\int e^{-i\vec{p}\cdot\vec{x}}\, \bar{v}_s(-\vec{p})\, \gamma^0 \psi(x) = b_s(-\vec{p})$$

$$\Longrightarrow \left[a_s(\vec{p}),\, a_r^\dagger(\vec{k}) \right] = (2\pi)^3 2\omega_p \delta_{rs} \delta^3(p-k)$$

$$a a^\dagger - a^\dagger a \quad \mathbb{1} > 0$$

$$\left[b_s(\vec{p}),\, b_r^\dagger(\vec{k}) \right] = (2\pi)^3 2\omega_p \delta_{rs} \delta^3(p-k)$$

$$H = \int d^3x\, \psi^+ h_D \psi(x) =$$

$$= \int d^3x \int d\Omega_p\, d\Omega_q\, e^{i(\vec{q}-\vec{p})\cdot\vec{x}} \sum_r \sum_s \left(u_r(p)\, a_r(p) + v_r(-p)\, b_r(-p) \right)^+ h_D \left(u_s(q)\, a_s(q) + v_s(-q)\, b_s(-q) \right)$$

$$= \int d\Omega_p\, \frac{1}{2\omega_p} \sum_{r,s} \left(u_r(p)\, a_r(p) + v_r(-p)\, b_r(-p) \right)^+ \omega_p \left(u_s(p)\, a_s(p) - v_s(-p)\, b_s(-p) \right)$$

$$= \int d\Omega_p\, \omega_p \sum_r a_r(p)^+ a_r(p) - b_r(p)\, b_r^+(p)$$

trivial constant

remember #6

$$= \int d\Omega_p\, \omega_p \sum_r \left[a_r^+(p)\, a_r(p) - b_r^+(p)\, b_r(p) \right] - \cancel{\left[b_r(p), b_r^+(p) \right]}$$

$$\int d\Omega\, \omega_p\, a^+ a$$

- a_r : $[H, a_r^\dagger(p)] = \omega_p a_r^\dagger(p)$
  $[H, a_r(p)] = -\omega_p a_r(p)$

  if $a_r(p)|0\rangle = 0 \ \forall \vec{p}$
  a-quanta $\Rightarrow$ $E > 0$

- b_r : $[H, b_r^\dagger(p)] = -\omega_p b_r^\dagger(p)$
  $[H, b_r(p)] = \omega_p b_r(p)$

  if $b_r(p)|0\rangle = 0 \ \forall \vec{p}$
  b-quanta $\Rightarrow$ $E < 0$

⚠ What about $|0\rangle$ such that $\forall \vec{p}$ $b_r^\dagger(p)|0\rangle = 0$?

$$\langle 0| [b_r(p), b_r^\dagger(k)] |0\rangle = (2\pi)^3 2\omega_p \delta^3(p-k) \langle 0|0\rangle$$

$$= -\langle 0| b_r^\dagger(k) b_r(p) |0\rangle = (2\pi)^3 2\omega_p \delta^3(p-k) \langle 0|0\rangle$$

$b_r(p)|0\rangle \equiv |\vec{p}\rangle$

$$-\langle \vec{k} | \vec{p} \rangle = (2\pi)^3 \, 2\omega_p \, \delta^3(p-k) \langle 0|0 \rangle$$

$$\underbrace{\langle \vec{k} | \vec{p} \rangle = -(2\pi)^3 \, 2\omega_p \, \delta^3(p-k) \langle 0|0 \rangle}$$

either $|p\rangle$ or $|0\rangle$ has <u>negative norm</u> !!

$$p$$

$$\langle \vec{k} | \vec{p} \rangle = -(2\pi)^3 2\omega_p \delta^3(p-k) \langle 0|0\rangle$$

$$|\varphi\rangle \equiv \int d\Omega_p \, f(\vec{p}) \, |\vec{p}\rangle$$

$$p$$

$$\int d\Omega_p d\Omega_k \langle \vec{k} | \vec{p} \rangle \left(f^*(k) f(p) \right) = \int -(2\pi)^3 2\omega_p \delta^3(p-k) \langle 0|0\rangle \underbrace{\quad}_{f(k)^* f(p)} d\Omega_p d\Omega_k$$

$$\Downarrow$$

$$\langle \varphi | \varphi \rangle = -\int |f(k)|^2 d\Omega_k \times \langle 0|0\rangle$$

$$[\psi, \psi^\dagger] = 1$$

⚠ Alternative route to quantization:

postulate anti-commutation relations

$$\psi_\alpha(x)\,\psi_\beta^+(y) + \psi_\beta^+(y)\,\psi_\alpha(x) \equiv \{\psi_\alpha(x), \psi_\beta^+(y)\} = \delta_{\alpha\beta}\,\delta^3(x-y)$$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\psi_\alpha^+(x), \psi_\beta^+(y)\} = 0$$

- $\dot{\psi} = i[H, \psi]$ ~ Dirac eq. ☺

- $\psi(x) = \int d\Omega_p \, e^{ip\cdot x} \sum_s a_s(\vec{p}) u_s(\vec{p}) + b_s(-\vec{p}) v_s(-\vec{p}) \Longrightarrow$

★ $\{a_s(p), a_r^+(k)\} = \{b_s(p), b_r^+(k)\} = (2\pi)^3 2\omega_p \delta^3(p-k)\delta_{rs}$

$\{a_s(p), a_r(k)\} = \{b_s(p), b_r(k)\} = \{a_s(p), b_r(k)\} = 0$

★ $H = \int d\Omega_p \; \omega_p \sum_r a_r^+(p) a_r(p) - b_r^+(p) b_r(p)$

$b_r(p) \equiv \tilde{b}_r^+(p) \qquad b_r^+(p) \equiv \tilde{b}_r(p)$

$$H = \int d\Omega_p \; \omega_p \sum_r a_r^\dagger(p)\, a_r(p) - \tilde{b}_r(p)\, \tilde{b}_r^\dagger(p)$$

$$H = \int d\Omega_p \; \omega_p \sum_r a_r^\dagger(p)\, a_r(p) + \tilde{b}_r^\dagger(p)\, \tilde{b}_r(p) \quad - \text{const.}$$

- postulate : $|0\rangle \;\; | \;\; a_r(p)|0\rangle = \tilde{b}_r(p)|0\rangle = 0$

$\Rightarrow$ all excited states have <u>positive energy</u>

- $[H, a_r^\dagger(p)] = \omega_p a_r^\dagger(p)$

  $[H, \tilde{b}_r^\dagger(p)] = \omega_p \tilde{b}_r^\dagger(p)$

both a and b quanta have positive energy !!!

$$H \sim \omega a^\dagger a$$

$$\underline{[H, a^\dagger]} = [\omega a^\dagger a, a^\dagger] = \omega a^\dagger \overbrace{\{a, a^\dagger\}}^{\mathbb{1}} - \omega \cancel{\{a^\dagger, a^\dagger\}} a$$

$$= \underline{\omega a^\dagger}$$

general result $\quad [BC, A] = \begin{cases} B[C, A] + [B, A]C \\ B\{C, A\} - \{B, A\}C \end{cases}$ check

• A toy example

$$b^2 = b^{+2} = 0$$

• single pair b, b^+: $\{b, b^+\} = \mathbb{1}$ $\quad \underline{\{b, b\}} = \underline{\{b^+, b^+\}} = 0$

• $|0\rangle$: $b|0\rangle = 0$

• $b^+|0\rangle \equiv |1\rangle$

• $b|1\rangle = bb^+|0\rangle = (\mathbb{1} - b^+b)|0\rangle = |0\rangle$

• $b^+|1\rangle = (b^+)^2|0\rangle = 0$

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$b^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

• $|0\rangle$ no quanta generated by b^+

• $|1\rangle$ 1-quanta is generated by b^+

• $|1\rangle$ no quanta generated by b

• $|0\rangle$ 1-quanta is generated by b

- $H = \omega b^+ b$

$= -\omega b b^+ + \text{const}$

breaks

$b \leftrightarrow b^+$ equivalence

- $H|0\rangle = 0$

$H|1\rangle = \omega|1\rangle$

$\omega > 0 \implies |0\rangle$ "vacuum"

$\omega < 0 \implies |1\rangle$ "vacuum"

- $b^{+2} = 0 \implies$ vacuum can be filled only once

$\equiv$ Exclusion Principle

Fock space

$\vert 0\rangle$	$a_r^\dagger(p)\vert 0\rangle$	$\tilde{b}_r^\dagger(p)\vert 0\rangle$	$a_r^\dagger(k)\, a_s^\dagger(p)\vert 0\rangle$	$\cdots$
$\vert 0\rangle$	$\vert p, r\rangle$	$\vert \widetilde{p, r}\rangle$	$\vert k, r; p, s\rangle$	$\cdots$
$E \rightarrow$ 0	ω_p	ω_p	$\omega_k + \omega_p$	

- two particle state:

$$|\psi\rangle = \int d\Omega_1\, d\Omega_2 \sum_{r_1, r_2} f(k_1, r_1; k_2, r_2)\, a^\dagger_{r_1}(k_1)\, a^\dagger_{r_2}(k_2)\, |0\rangle$$

$$= \int d\Omega_1 \, d\Omega_2 \; - \sum_{r_1, r_2} f(k_1, r_1; k_2, r_2) \, a^+_{r_2}(k_2) \, a^+_{r_1}(k_1) |0\rangle$$

$$k_1 \leftrightarrow k_2 \qquad r_1 \leftrightarrow r_2$$

$$= -\int d\Omega_1 \, d\Omega_2 \sum_{r_1, r_2} f(k_2, r_2; k_1, r_1) \, a^+_{r_1}(k_1) \, a^+_{r_2}(k_2) |0\rangle$$

$$\boxed{\Longrightarrow \; f(k_1, r_1; k_2, r_2) = - f(k_2, r_2; k_1, r_1)}$$

$$a^+_{r_1}(k_1) \, a^+_{r_2}(k_2) = |k_1 r_1; k_2 r_2\rangle = - |k_2 r_2; k_1 r_1\rangle$$

a and is quanta follow

Fermi-Dirac Statistics

$\psi \sim (j_-, j_+)$

$\vec{J} = \vec{J}_+ + \vec{J}_-$

$j_- + j_+ = \text{integer} \Rightarrow BE$

$j_- + j_+ = \text{half-integer} \Rightarrow FD$

Spin-statistics theorem

$\psi\left(\frac{1}{2}, \frac{1}{2}\right) \longrightarrow \frac{1}{2} + \frac{1}{2} = 1 \quad BE$

Other charges

- Notation $\tilde{b}_r(p) \longrightarrow b_r(p)$

$$\psi = \int d\Omega_p \; e^{ip\cdot x} \sum_r a_r(p)\, u_r(p) + b_r^\dagger(-p)\, v_r(-p)$$

- $$\vec{P} = -i \int d^3x \; \psi^\dagger \vec{\nabla} \psi$$

$$= \int d\Omega_p \; \vec{p} \sum_r a_r^\dagger(p)\, a_r(p) + b_r^\dagger(p)\, b_r(p)$$

- $$P^\mu = \int d\Omega_p \; p^\mu \sum_r a_r^\dagger(p)\, a_r(p) + b_r^\dagger(p)\, b_r(p)$$

$\Rightarrow$ 4-types of quanta with same mass m

$$a_1^+ , \quad a_2^+ , \quad b_1^+ , \quad b_2^+$$

$$-i\int \psi^\dagger \vec{\nabla} \psi = \int d\Omega_p \; \frac{1}{2\omega_p} \cdot$$

$$\sum_{r,s} \left(u_r^\dagger(p)\, a_r^\dagger(p) + v_r^\dagger(-p)\, b_r(-p) \right) \vec{p} \left(u_s(p)\, a_s(p) + v_s(-p)\, b_s^\dagger(-p) \right)$$

$$= \int d\Omega_p \; \vec{p} \sum_r a_r^\dagger(p)\, a_r(p) + b_r(-p)\, b_r^\dagger(-p)$$

$$= \int d\Omega_p \; \vec{p} \sum_r a_r^\dagger(p)\, a_r(p) - b_r^\dagger(-\vec{p})\, b_r(-\vec{p})$$

$$= \int d\Omega_p \; \vec{p} \sum_r a_r^\dagger(p)\, a_r(p) + b_r^\dagger(\vec{p})\, b_r(\vec{p})$$

- Phase rotation symmetry $\qquad \psi \to e^{-i\alpha}\psi$

$$\bar{\psi} \to e^{i\alpha}\bar{\psi}$$

- Noether current: $\qquad J^\mu = \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\delta\psi = i\bar{\psi}\gamma^\mu(-i\psi)$

$$= \bar{\psi}\gamma^\mu\psi$$

- $Q = \int d^3x \, J^0$

$$= \int d^3x \, \psi^\dagger \psi =$$

$$= \int \frac{d\Omega_p}{2\omega_p} \cdot \sum_{r,s} \left(u_r^\dagger(p)\, a_r^\dagger(p) + v_r^\dagger(-p)\, b_r(-p)\right)\left(u_s(p)\, a_s(p) + v_s(-p)\, b_s^\dagger(-p)\right)$$

$$= \int d\Omega_p \sum_r a_r^+(p)\, a_r(p) + b_r(p)\, b_r^+(p)$$

$$= \int d\Omega_p \sum_r a_r^+(p)\, a_r(p) - b_r^+(p)\, b_r(p) + \text{const.}$$

- $a_r^+(p) \to$ quanta with $Q = 1$ fermion
- $b_r^+(p) \to$ quanta with $Q = -1$ anti-fermion

△ Heisenberg Picture

$$e^{iHt} a_r(p) e^{-iHt} = e^{-i\omega_p t} a_r(p)$$

$$e^{iHt} b_r(p) e^{-iHt} = e^{-i\omega_p t} b_r(p)$$

$$\psi(x) = e^{iHt} \psi(x) e^{-iHt} \qquad x = (t, \vec{x})$$

$$= \int d\Omega_p \; e^{i\vec{p}\cdot\vec{x}} \sum_r e^{-i\omega_p t} u_r(p) a_r(p) + e^{i\omega_p t} v_r(-\bar{p}) b_r^{+}(-p)$$

$$= \int d\Omega_p \sum_r e^{-ipx} u_r(p) a_r(p) + e^{ipx} v_r(\vec{p}) b_r^{+}(\vec{p})$$

Angular Momentum & Boosts

$$\varphi(x) \longrightarrow \varphi(\Lambda^{-1}x)$$

$$\psi_\alpha(x) \longrightarrow \underbrace{D_{\alpha\beta}(\Lambda)}\,\psi_\beta(\Lambda^{-1}x)$$

$$= \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix} \equiv \Lambda_D$$

$$\Lambda_D = e^{-i\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}$$

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$\left[\begin{array}{l} S^{0i} = -\frac{i}{2}\begin{bmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{bmatrix} \\ \\ S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix} \end{array}\right.$$

$$\psi(x) \longrightarrow e^{-\frac{i\omega_{\mu\nu}}{2} S^{\mu\nu}} \psi(\Lambda^{-1}x)$$

$$\Delta\psi = \omega_{\mu\nu}\left(-i\frac{S^{\mu\nu}}{2} + \frac{1}{2}\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)\right)\psi$$

$$J^\rho_{\mu\nu} \Rightarrow J_{\mu\nu} \equiv \int J^0_{\mu\nu}\, d^3x$$

$$J_{\mu\nu} \begin{cases} J_{0i} \\ J_{ij} = \frac{1}{2}\,\varepsilon_{ijk}\,J_k \end{cases}$$

$\rightarrow$ angular momentum

<u>Exercise</u>

$$J_k = \int d^3x\, \psi^\dagger \Big[\underbrace{\vec{x} \wedge (-\vec{\nabla})} + \underbrace{\frac{\vec{\Sigma}}{2}} \Big] \psi$$

$$\vec{J} = \vec{L} + \vec{S}$$

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$